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The Eshelby tensor☆

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ABSTRACT

The problem of the existence of a tensor that is inverse to the well-known Eshelby tensor, which connects the free homogeneous and hindered strains of an ellipsoidal elastic inclusion undergoing transformation, is investigated. It is shown that this tensor exists for inclusions in the form of oblate and prolate spheroids in isotropic elastic space. Certain applications are considered, in particular problems of determining the stresses in ellipsoidal rigid and rigid plastic inclusions.

The classic results obtained by Eshelby[1] relate to ellipsoidal elastic inclusions in an elastic medium subjected to two types of effects:

1. Changes in the shape and linear dimensions, which, in the absence of resistance of the surrounding medium, correspond to homogeneous strain; there are no external forces in this case.
2. The inclusion and medium have different elastic constants; evenly distributed stresses are applied at infinity.

The relations between hindered and free homogeneous strains (case 1) and between strains at infinity and in the inclusion (case 2) will contain an asymmetrical fourth-rank tensor. The same tensor occurs in relations linking stresses and strains at infinitely remote points of the elastic medium and in an ellipsoidal physically non-linear inclusion contained in this medium.[2,3] Note that in all these cases the stress–strain state in the inclusion will be homogeneous.

A natural question arises: is there an inverse tensor to the Eshelby tensor? This will unable us to invert the above-mentioned relations between the hindered and free strains of the inclusion, and also to consider new problems, for example, the determination of the stresses in ellipsoidal rigid and rigid-plastic inclusions.

The problem of the inverse Eshelby tensor in the general case of an ellipsoid with different semi-axes proved to be non-trivial. Its existence for ellipsoids of rotation in isotropic elastic space is shown below.

1. The existence of the inverse Eshelby tensor for inclusions in the form of oblate and prolate spheroid

We will consider an elastic space ν containing an ellipsoidal inclusion $\upsilon*$, the boundary equation of which in the selected coordinate system x_1, x_2, x_3 has the form $x_k^2 a_k^{-2} = 1 (a_1 \geq a_2 \geq a_3)$. Here and below, summation from 1 to 3 over repeated indices is implied unless stated otherwise.

As has been shown,[1] the tensor $\mathbf{S}$ (the Eshelby tensor) is independent of x_k ($k = 1, 2, 3$) but is determined by the geometry of the region $\upsilon*$ and the characteristics of the medium υ. In the case of an isotropic elastic space υ, non-zero components of the given tensor are determined in the following way[1–3] (ν is Poisson's ratio)

$$S_{kkkk} = Qa_k^2 I_{kk} + RI_k, \quad S_{kkll} = Qa_l^2 I_{kl} - RI_k$$

$$2S_{klkl} = 2S_{kllk} = Q(a_k^2 + a_l^2) I_{kl} + R(I_k + I_l)$$

$$Q = 3/[8\pi(1 - \nu)], \quad R = (1 - 2\nu)/[8\pi(1 - \nu)] \tag{1.1}$$

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$$I_k = 2\pi a_1 a_2 a_3 \int_0^\infty \frac{du}{(a_k^2+u)\Delta}, \quad I_{kk} = 2\pi a_1 a_2 a_3 \int_0^\infty \frac{du}{(a_k^2+u)^2\Delta}, \quad I_{kl} = \frac{2}{3}\pi a_1 a_2 a_3 \int_0^\infty \frac{du}{(a_k^2+u)(a_l^2+u)\Delta}$$

$$\Delta^2 = (a_1^2+u)(a_2^2+u)(a_3^2+u)$$

($k, l = 1, 2, 3$; $k \neq l$; no summation with respect to k and l); the remaining components $S_{klmn} = 0$.

The quantities I_k, I_{kk} and I_{kl} are expressed in terms of elliptic integrals of the first and second kind and can be found if any two of the quantities I_k are known, as, for example, for oblate and prolate spheroids, where I_k represents the elementary functions a_1, a_2, a_3, and the following equations will occur ($\delta < 1$):when $a_1 = a_2 = \alpha$ and $a_3 = \delta\alpha$

$$I_1 = I_2 = 2\pi\delta(1-\delta^2)^{-3/2}[\arccos\delta - \delta(1-\delta^2)^{1/2}], I_3 = 4\pi - 2I_1$$

$$I_{11} = I_{22} = 3I_{12} = \frac{3I_1 - 4\pi\delta^2}{4\alpha^2(1-\delta^2)}, \quad I_{13} = I_{23} = \frac{4\pi - 3I_1}{3\alpha^2(1-\delta^2)}, \quad I_{33} = \frac{4\pi(1-3\delta^2)+6I_1\delta^2}{3\alpha^2\delta^2(1-\delta^2)} \tag{1.2}$$

when $a_1 = \alpha$ and $a_2 = a_3 = \delta\alpha$

$$I_1 = 4\pi - 2I_2, \quad I_2 = I_3 = \frac{2\pi}{\delta}\left(\frac{1}{\delta^2}-1\right)^{-3/2}\left[\frac{1}{\delta}\left(\frac{1}{\delta^2}-1\right)^{1/2} - \operatorname{arch}\frac{1}{\delta}\right]$$

$$I_{11} = \frac{4\pi(3-\delta^2)-6I_2}{3\alpha^2(1-\delta^2)}, \quad I_{22} = I_{33} = 3I_{23} = \frac{4\pi - 3I_2\delta^2}{4\alpha^2\delta^2(1-\delta^2)}, \quad I_{12} = I_{13} = \frac{3I_2 - 4\pi}{3\alpha^2(1-\delta^2)} \tag{1.3}$$

The 6×6 matrix $\|s_{kl}\|$, the elements s_{kl} of which are defined in the following way:[3] $s_{kl} = S_{kkll}$ ($k, l = 1, 2, 3$; no summation with respect to k and l), $s_{44} = 2S_{1212}$, $s_{55} = 2S_{1313}$, $s_{66} = 2S_{2323}$ and the remaining s_{kl} are equal to zero, corresponds to the tensor S.

Finding the inverse tensor $\mathbf{S}^{-1}$ is equivalent to finding the inverse matrix to $\|s_{kl}\|$, for which, as follows from the above formulae for s_{kl} ($k, l = 1, 2, 3$) and the inequalities $s_{kk} > 0$ ($k = 4, 5, 6$; no summation with respect to k), it is necessary and sufficient for the 3×3 matrix equal to $\|s_{kl}^{\text{o}}\| \equiv \|s_{kl}\|$ ($k, l = 1, 2, 3$) to be non-degenerate, i.e.,

$$\Delta_0 \equiv \det\|s_{kl}^{\text{o}}\| \neq 0 \tag{1.4}$$

We will show that, for the above-mentioned ellipsoids of rotation, condition (1.4) is satisfied.

An inclusion in the form of an oblate spheroid. From relations (1.1) and (1.2), omitting the calculations, we obtain

$$\Delta_0 = \frac{1+\nu}{32\pi^2(1-\nu)^3(1-\delta^2)}\left[\frac{3I-4\pi\delta^2}{4(1-\delta^2)} + (1-2\nu)I\right]\left\{[3-4\nu(1-\delta^2)]I - 4\pi\delta^2 - (1-2\nu)(1-\delta^2)\frac{I^2}{\pi}\right\} \tag{1.5}$$

From relations (1.2) and (1.5) it can be seen that Δ_0 is the function of $\delta (0 < \delta < 1)$ and $3I > 4\pi\delta^2$. Introducing the new variable $t = 1 - \delta^2$ ($0 < t < 1$), we conclude that the condition $\Delta_0 = 0$ is equivalent to the equation

$$\Psi(t) \equiv 2(1-2\nu)tF^2 + (4\nu t - 3)F + 2(1-t) = 0$$
$$F(t) \equiv I/(2\pi) = (1-t)^{1/2}t^{-3/2}[\arccos(1-t)^{1/2} - (1-t)^{1/2}t^{1/2}] \tag{1.6}$$

We will consider the case of an incompressible elastic medium when $\nu = 1/2$. From Eq. (1.6) we find

$$f_1(t) \equiv \arccos(1-t)^{1/2} = f_2(t) \equiv 3(t-t^2)(3-2t)^{-1} \tag{1.7}$$

Equation (1.7) in t, for $0 < t < 1$, has no roots, since

$$f_1(0) = f_2(0) = 0, \quad f_1'(t) = \frac{1}{2}(t-t^2)^{-1/2} > f_2'(t) = \frac{9/2 - 6t}{(3-2t)^2(t-t^2)^{1/2}}$$

From this it follows that $f_1(t) > f_2(t)$.

Thus, Eq. (1.6) with $\nu = 1/2$ has no solutions; consequently, $\Delta_0 \neq 0$.

Let $0 \leq \nu < 1/2$. Considering Eq. (1.6) as an equation in F, we obtain its roots

$$F_{1,2} = \frac{3 - 4\nu t \pm \sqrt{D}}{4(1-2\nu)t}; \quad D = 16(1-\nu)^2t^2 - 8(2-\nu)t + 9 > 0 \tag{1.8}$$

The inequality occurs by virtue of the fact that

$$\min_{0<t<1} D(t) = (1-2\nu)(5-4\nu)(1-\nu)^{-2} > 0$$

For the function $\psi(t)$ of system (1.6) we obtain

$$\Psi(t) = 2(1-2\nu)t(F-F_1)(F-F_2) \tag{1.9}$$

Since $I_k > 0$ (k, l = 1, 2, 3), from Eqs. (1.2) it follows that $2(1-t)/3 < F < 2/3$, and from Eqs. (1.8) we have

$$F_1 > \frac{3-4\nu t}{4(1-2\nu)t} > \frac{2}{3}$$

(the last inequality follows from the fact that $9-8t+4\nu t > 0$). Therefore, $F-F_1 < 0$, and, in view of Eq. (1.9), the condition $\psi(t)$=0 is equivalent to $F = F_2$.

We will show that this equation in t has no roots for $0 < t < 1$.

In fact, the inequality $f_1(t) > f_2(t)$ established above for the functions from (1.7) is equivalent to the following:

$$F > F_2 \equiv \frac{3-4\nu t-\sqrt{D}}{4(1-2\nu)t} = \frac{4(1-t)}{3-4\nu t+\sqrt{D}} \quad \text{при} \quad \nu \to \frac{1}{2} \tag{1.10}$$

(F, F_2 and D are defined by formulae (1.6) and (1.8)).

Considering F_2 as a function of ν, we will find its derivative:

$$F_2'(\nu) = \frac{16(1-t)t}{(3-4\nu t+\sqrt{D})^2}\left[1+\frac{4(1-\nu)t-1}{\sqrt{D}}\right]$$

From this it can be seen that, if $f_3 \equiv 4(1-\nu)t-1 \geq 0$, then $F_2'(\nu) > 0$.

If $f_3 < 0$, again we will have $F_2'(\nu) > 0$, since $D-(-f_3)^2 = 8(1-t) > 0$.

Consequently, $F_2 = F_2(\nu)$ is an increasing function, and from the inequality (1.10) we obtain

$$F > F_2(\nu = 1/2) > F_2(\nu < 1/2)$$

Then, taking into account that $F-F_1 < 0$, from relations (1.8) and (1.9) we find $\psi(t) < 0$ when $0 < t < 1$ and $0 \leq \nu \leq 1/2$.

An inclusion in the form of a prolate spheroid. In this case, from relations (1.1) and (1.3), carrying out calculations similar to the previous ones, we can establish that the condition $\det \left\| s_{kl}^{o} \right\| = 0$ is equivalent to the equation

$$\Psi^{\circ}(t) \equiv 2(1-2\nu)t(F^{\circ})^2 + [(4\nu-3)t+3]F^{\circ} - 2 = 0 \tag{1.11}$$

$$F^{\circ} \equiv \frac{1}{t} - \frac{1-t}{t^{3/2}}\ln\frac{1+t^{1/2}}{(1-t)^{1/2}}$$

When $\nu = 1/2$, we will have

$$f_1^{\circ}(t) = \ln\frac{1+t^{1/2}}{(1-t)^{1/2}} = f_2^{\circ}(t) \equiv \frac{3t^{1/2}}{3-t} \tag{1.12}$$

Equation (1.12), as with Eq. (1.7), has no roots for $0 < t < 1$, since $f_1^{o}(0) = f_2^{o}(0)$ and

$$(f_1^{\circ}(t))' = \frac{1+t^{1/2}}{2(1+t^{1/2})(1-t)} > (f_2^{\circ}(t))' = \frac{3}{2}\frac{3t^{-1/2}+t^{1/2}}{(3-t)^2}.$$

Consequently, $f_1^{o}(t) > f_2^{o}(t)$.

When $0 \leq \nu < 1/2$, from equality (1.11), by analogy with Eq. (1.9), we obtain

$$\Psi^{\circ}(t) = 2(1-2\nu)t(F^{\circ}-F_1^{\circ})(F^{\circ}-F_2^{\circ})$$

$$F_1^{\circ} \equiv \frac{(3-4\nu)t-3+\sqrt{D_0}}{4(1-2\nu)t} = \frac{4}{3-(3-4\nu)t+\sqrt{D_0}}$$

$$F_2^{\circ} = \frac{(3-4\nu)t-3-\sqrt{D_0}}{4(1-2\nu)t} < 0; \quad D_0 = (3-4\nu)^2t^2 - 2(1+4\nu)t + 9 > 0 \tag{1.13}$$

The condition $\psi^{o}(t)$=0 reduces to the equation $F^{o} = F_1^{o}$, which has no solutions when $0 < t < 1$. In fact, the inequality $f_1^{o}(t) > f_2^{o}(t)$ for functions (1.12) is equivalent to $F^{o} = F_1^{o}$ when $\nu \to 1/2$. For the derivative of the function F_1^{o} with respect to ν we have

$$(F_1^{\circ}(\nu))' = \frac{16t}{[3-(3-4\nu)t+\sqrt{D_0}]^2}\left[\frac{1+(3-4\nu)t}{\sqrt{D_0}}-1\right] < 0,$$

since $[1+(3-4\nu)t]^2 - D_0 = 8(t-1) < 0$. Consequently, $F_1^\circ = F_1^\circ(\nu)$ is a decreasing function, and therefore

$$F^\circ < F_1^\circ(\nu = 1/2) < F_1^\circ(\nu < 1/2)$$

and from relations (1.13) we find $\psi^\circ(t)<0$ when $0<t<1$ and $0 \le \nu \le 1/2$.

2. Determination of the stresses in rigid inclusions

We will examine an elastic space with an ellipsoidal rigid inclusion ν^* subjected to the action of stresses σ_{kl}^∞ ($k, l = 1, 2, 3$) uniformly distributed at infinity. The problem arises as to the possibility of determining the stresses in the region $\upsilon*$.

Earlier[2,3] a similar problem was investigated for an ellipsoidal physically non-linear inclusion with constitutive equations of fairly general form

$$\varepsilon_{kl}^* = F_{kl}(\sigma_{mn}^*),\ \sigma_{kl}^* = G_{kl}(\varepsilon_{mn}^*);\quad k, l, m, n = 1, 2, 3$$

where F_{kl} and G_{kl} are the components of mutually inverse tensor operators. In region υ, Hooke's law holds, namely,

$$\varepsilon_{kl} = a_{klmn}\sigma_{mn},\quad k, l = 1, 2, 3$$

where a_{klmn} are components of the elastic compliance tensor.

The following relations were established[2,3] between the stress–strain states in the inclusion and at infinity

$$\varepsilon_{kl}^* = \varepsilon_{kl}^\infty + S_{klmn}(\varepsilon_{mn}^* - \tilde{\varepsilon}_{mn}^*),\quad \varepsilon_{kl}^\infty = a_{klmn}\sigma_{mn}^\infty,\quad \tilde{\varepsilon}_{kl}^* \equiv a_{klmn}\sigma_{mn}^* \tag{2.1}$$

where S_{klmn} are the components of the tensor **S**. For the rigid inclusion $\upsilon*$ examined here, when $\varepsilon_{kl}^* = 0$ ($k, l = 1, 2, 3$) we obtain

$$S_{klmn}\tilde{\varepsilon}_{mn}^* = \varepsilon_{kl}^\infty \tag{2.2}$$

Then (S_{klmn}^{-1} are the components of the tensor $\mathbf{S}^{-1}$)

$$\tilde{\varepsilon}_{kl}^* = S_{klmn}^{-1}\varepsilon_{mn}^\infty \tag{2.3}$$

From relations (2.1) to (2.3) it can be seen that the components of the stresses σ_{kl}^* will be found if condition (1.4) is satisfied.

If the inclusion $\upsilon*$ is rigid plastic, i.e., $\varepsilon_{kl}^* = 0$ when $s < \sigma_T$, where σ_T is the yield point, and $s = s(\sigma_{kl}^*)$ is a first-degree homogeneous function (the equivalent stress, for example the stress intensity or the maximum shear stress), and the loading at infinity is simple: $\sigma_{klo}^\infty = \sigma_{klo}^\infty\tau$, where $\tau \ge 0$ is the loading parameter, then, according to Eq. (2.3), we have $\sigma_{kl}^* = \sigma_{klo}^*\tau$. Hence, $s(\sigma_{kl}^*) = s(\sigma_{klo}^*)\tau$, and it is possible to find the value τ_T of the parameter τ for which the inclusion $\upsilon*$ will transfer into the plastic state when $s = \sigma_T : \tau_T = \sigma_T/s(\sigma_{klo}^*)$.

As another example we can examine a heteromodular inclusion whose strains depend on the sign of the first invariant $I_\sigma^* = \sigma_{kk}^*$ of the stress tensor.

A model of an isotropic heteromodular elastic material whose bulk modulus depends on the sign of I_σ^* has been proposed.[4] It can be extended to the case of a rigid inclusion in the following way. We will assume that, when $I_\sigma^* \ge 0$, the region $\upsilon*$ is elastic and has the same characteristics as the surrounding medium υ, and that, when $I_\sigma^* < 0$, it is undeformable, i.e., $\varepsilon_{kl}^* = 0$ ($k, l = 1, 2, 3$). Then, under specified external stresses σ_{kl}^∞, from Eq. (2.1) we will have $\varepsilon_{kl}^* = \varepsilon_{kl}^\infty$ when $I_\sigma^\infty \ge 0$, as $\varepsilon_{kl}^* = \tilde{\varepsilon}_{kl}^* = a_{klmn}\sigma_{mn}^*$. If $I_\sigma^\infty < 0$, then likewise $I_\sigma^* < 0$, as, in the opposite case, i.e. when $I_\sigma^* \ge 0$, from Eq. (2.1) it would follow that $I_\sigma^\infty = I_\sigma^* \ge 0$ in view of the fact that $\varepsilon_{kl}^\infty = \varepsilon_{kl}^*$. Then, from Eq. (2.1) with $\varepsilon_{kl}^* = 0$ we will have the relations (2.2) and (2.3) for finding the stresses in a rigid inclusion (with $I_\sigma^* < 0$).

3. Conclusions

In the cases examined above of inclusions in the form of ellipsoids of revolution it has been shown that condition (1.4), which ensures the existence of an inverse Eshelby tensor, is satisfied. This enables us to invert the above-mentioned relations between hindered and free strains examined by Eshelby,[1] guarantees the singularity of the solution of problems of determining the stresses in ellipsoidal rigid inclusions and also makes it possible to determine the instant when rigid plastic inclusions transfer into the plastic state under the action of external monotonically increasing stresses.

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